

Holography, Padé approximants and deconstruction

Adam Falkowski^{ab} and Manuel Pérez-Victoria^a

^a *CERN Theory Division,
CH-1211 Geneva 23, Switzerland*

^b *Institute of Theoretical Physics, Warsaw University,
Hoża 69, 00-681 Warsaw, Poland*

E-mail: adam.falkowski@cern.ch, mpv@cern.ch

ABSTRACT: We investigate the relation between holographic calculations in 5D and the Migdal approach to correlation functions in large- N_c theories. The latter employs Padé approximation to extrapolate short-distance correlation functions to large distances. We make the Migdal/5D relation more precise by quantifying the correspondence between Padé approximation and the background and boundary conditions in 5D. We also establish a connection between the Migdal approach and the models of deconstructed dimensions.

KEYWORDS: Large Extra Dimensions, AdS-CFT Correspondence, 1/N Expansion, Phenomenological Models.

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1. Introduction

Holography relates strongly-coupled gauge theories to weakly-coupled theories in higher dimensions. The original conjecture [1] connects type IIB string theory in the gravitational $\text{AdS}_5 \times S^5$ background to the 4D $\mathcal{N} = 4$ $\text{SU}(N_c)$ superconformal field theory. This correspondence can be extended to other asymptotically-AdS [2] spaces, and examples of geometries in which conformal invariance is broken in the IR are known (see, for instance, [3] and references therein). From a phenomenological point of view, it is often sufficient to consider the simpler relation between non-supersymmetric 5D field theories in a Randall-Sundrum [4] background and 4D large- N_c strongly-coupled theories with conformal invariance spontaneously broken [5]. This can be applied, for example, in studying properties of QCD at large N_c [6] or to clarify the physics of electroweak symmetry breaking by strong dynamics [7].

The 5D setup involves a slice of AdS_5 truncated by two branes at $z = z_{\text{UV}}$ and $z = z_{\text{IR}}$. The bulk hosts 5D gauge and, possibly, other spin fields, which represent composite operators of the dual CFT. One way to probe the dynamics of such theories is to compute

UV boundary correlators of the 5D bulk fields. For example, the 1PI two-point correlation function of boundary gauge fields is given by the expression

$$\int d^4x e^{-ipx} \langle A_\mu(x) A_\mu(0) \rangle_{1\text{PI}} = \left(-\eta_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} \right) \Pi(p^2), \quad \Pi(p) \sim \left. \frac{\partial_z G(z, p)}{G(z, p)} \right|_{z=z_{\text{UV}}}, \quad (1.1)$$

where $G(z, p)$ is a solution of the 5D equations of motion subject to appropriate boundary conditions at the IR brane. The holographic dictionary relates this boundary correlator to the connected two-point correlation function of a conserved global symmetry current of the 4D CFT. The poles of this correlation function are interpreted as resonances of the CFT.

There exists another, seemingly unrelated approach to computing correlation functions in 4D strongly coupled large- N_c theories, which was proposed long ago by Migdal [8]. The program of Migdal aims at reproducing the gauge theory correlators at low p^2 using information about the deep Euclidean regime. The main input is the non-analytic behaviour of the correlation functions at large Euclidean momenta, where the correlators, at leading order, exhibit a conformal behaviour, $\lim_{p^2 \rightarrow -\infty} \Pi(p) \equiv f(p^2) \sim -p^{2n} \log(-p^2)$. This asymptotic expression is approximated by a ratio of two polynomials of degree N , $f(p^2) \approx R_N(p^2)/S_N(p^2)$, by means of a Padé approximation. Finally, the result is analytically continued to low time-like p^2 and the large- N limit is taken.

It was recently pointed out by Shifman [9] and Voloshin, and shown in detail by Erlich et al. [10], that Migdal's procedure gives results similar to those of 5D holographic computations. Indeed, performing a series of Padé approximations on the input function $f(p^2) = -\log(-p^2/\mu^2)$, one obtains [10], for large N ,

$$f(p^2) \rightarrow \frac{-\log(p^2/\mu^2) J_0(2Np/\mu) + \pi Y_0(2Np/\mu)}{J_0(2Np/\mu)} \quad (1.2)$$

The same result would be obtained if we performed a computation of the boundary gauge field correlator in the Randall-Sundrum spacetime, with $z_{\text{IR}} = 2N/\mu$ and $z_{\text{UV}} = 1/\mu \rightarrow 0$. In fact, the similarity of the two approaches is not restricted to AdS₅. For example, Padé approximations of the input function $f(p^2) = (-p^2/\mu^2)^{-1/2}$ lead to a result that again coincides with the 5D boundary gauge correlator, but this time computed in the 5D Minkowski background.¹

This coincidence seems quite mysterious, as the Migdal approach never makes any reference to extra dimensions. Of course, the main ingredient of both methods, and the one that allows us to make contact with a large- N_c theory, is that the correlation functions are meromorphic. But this does not determine them uniquely. Actually, the success of the Migdal approach and its relation to holographic calculations raises a number of questions:

- i) Why, in the first place, can the series of Padé approximants be interpreted as physical correlation functions of a large- N_c gauge theory? The physical interpretation of the results obtained by Migdal is possible because the Padé approximants have only simple poles, with negative residues on the positive real p^2 axis. This is not a generic feature of Padé approximants.

¹This was first observed by Gherghetta, Pomarol and Rattazzi [11].

- ii) Why, in the large- N limit, do we recover correlators characteristic of theories in 5D? The distinguishing feature of 5D models is locality in the fifth coordinate. How is this locality encoded in the Migdal approach?
- iii) Which 5D geometries and which IR boundary conditions can be reproduced by the Migdal approach?
- iv) Does the Migdal procedure at finite N also correspond to some physical setup? There is a tantalizing similarity between the correlators computed in the Migdal approach and those obtained in deconstruction [12]. Both express the polarization function as a ratio of two polynomials in p^2 , which converges to a non-analytic function for large N and large Euclidean momentum (see [13] for a computation in deconstructed AdS₅). Is there a precise quantitative connection between the two approaches?

In this paper we provide answers to these questions. The first one was, in fact, already addressed in Migdal’s original papers [8]. It turns out that the nice physical properties of Padé approximants arise because the input function belongs to the class of so-called Stieltjes functions. In the mathematical literature, Padé approximants of Stieltjes functions have been extensively studied, and a connection to orthogonal polynomials has been established. The physical properties of Padé approximants are intimately related to certain familiar properties of orthogonal polynomials.

One well-known property of orthogonal polynomials is that they satisfy a second-order recursion relation in the polynomial degree: $\pi_{N+1}(w) = (a_n w + b_n)\pi_N(w) - c_n\pi_{N-1}(w)$. We show that the connection between Padé approximation and orthogonal polynomials implies that the polynomials entering the Padé approximants also satisfy a second-order recursion relation:

$$T_{N+1}(p^2) = A_N(p^2)T_N(p^2) - B_N(p^2)T_{N-1}(p^2), \tag{1.3}$$

where $T = R, S$. Hence, Padé approximants implement automatically a form of locality in the discrete space of the polynomial degrees. For large N , this recursion relation reduces to a second order differential equation, whose form is analogous to equations of motion in 5D theories. We discuss the necessary condition for the large- N limit of the Migdal approach to correspond to sensible 5D setups and find which geometries and which boundary conditions can be matched. We shall see that the precise manner in which the limit is taken is important.

Finally, we study the relation between Migdal’s approach at finite N and deconstruction. Deconstruction is a four-dimensional framework that involves a product gauge group G^N and a set of bifundamental non-linear sigma model fields (the links) [12]. Deconstruction models are parametrized by a set of gauge couplings g_j and decay constants v_j . For large N such setup is related to a 5D gauge theory with the gauge group G , where the fifth dimension is latticized. Each choice of g_j and v_j on the deconstruction side corresponds to some 5D warped geometry, and the dictionary between the two frameworks has been established [14]. In this paper we quantify the relation between deconstruction and the Migdal approach. We show that, given the coefficient of the recursion relation for

Padé approximants, we can identify the deconstruction parameters that yield the same polarization function as the Migdal approach. An unexpected result is that deconstruction models directly related to the Migdal approach are non-minimal; they must include the kinetic mixing between neighbouring gauge fields from the product group. This, however, corresponds to an irrelevant operator in the continuum limit.

The paper is organized as follows. In section 2 we present several examples that illustrate the connection of the Migdal approach with 5D theories. In section 3 we review the mathematical results connecting Padé approximants of Stieltjes functions to orthogonal polynomials. The large- N limit of Padé approximants of Stieltjes functions is studied in section 4 and the relation to 5D theories is quantified in section 5. In section 6 we discuss the connection of Migdal approximation at finite N to 4D deconstruction models. Section 7 contains our conclusions, and in the appendix we derive the holographic formula for two-point functions in deconstruction.

2. Migdal’s approximation in examples

For two-point correlation functions, the deep Euclidean limit $p^2 \rightarrow -\infty$ is a function $f(t)$ depending on a variable $t = p^2/\mu^2$, where μ is an arbitrary renormalization scale. The function $f(t)$ has a branch cut along the positive real axis and contains a perturbative piece plus power corrections induced by the condensates. We will be interested in the perturbative part only. Migdal proposes to compute Padé approximants of $f(t)$ at some Euclidean point $-\lambda < 0$. The Padé approximant is given by a ratio of two polynomials of degrees M and N , such that its Taylor expansion around $t = -\lambda$ matches that of $f(t)$ to order $(t + \lambda)^{M+N+1}$. Under certain assumptions, this series of approximants converges to $f(t)$ when $M, N \rightarrow \infty$. The idea of Migdal is to take instead a combined limit $N \rightarrow \infty$, $t \rightarrow 0$, keeping $\tilde{t} = tN^2$ and $M - N$ fixed. For finite momentum p , this amounts to sending N and μ to infinity with $\tilde{\mu} = \mu/N$ fixed. We call this *the Migdal limit*. In this limit, the spacing between the poles of the Padé approximants is controlled by the scale $\tilde{\mu}$ introduced in the limiting process. Intuitively, even though the spacing between poles goes to zero at large N , so as to reproduce the branch cut, zooming in the small t region simultaneously allows to resolve them. It turns out that, for the functions $f(t)$ of interest, the limiting expression has only simple poles located at timelike momenta and with negative residues. Therefore, the Migdal limit of $f(t)$ can be interpreted as the complete two-point function in the large N_c limit. By construction, this function has the correct deep Euclidean behaviour.

In this section we present several examples of the Migdal procedure. The details of the calculations are postponed until section 3, where we present a generalized approach to this kind of computation.

We start with the example discussed in [10]. The two-point correlation function of two conserved vector currents has the general form

$$\Pi_{\mu\nu}(p) = \left(\frac{p_\mu p_\nu}{p^2} - \eta_{\mu\nu} \right) \Pi(p^2). \tag{2.1}$$

For the vector and axial currents in QCD, the leading perturbative contribution to $\Pi(p^2)$ at large Euclidean momentum $p^2 < 0$ is proportional to $-p^2 \log(-p^2/\mu^2)$.² Ignoring multiplicative constants, we take $f(t+1) = -\log(-t)$. Next, we approximate $f(t+1)$ by $\Pi_N = t[N/N]_f$, where $[M/N]_f = [N+J/N]_f = R_N^J/S_N^J$ is the Padé approximant to $f(t+1)$ at $t = -1$, with R_N^J and S_N^J polynomials in t of degree $M = N+J$ and N , respectively. These polynomials can be determined to be

$$R_N^0(t) = (t+1)^N \bar{P}_N \left(\frac{1-t}{1+t} \right), \quad S_N^0(t) = (t+1)^N P_N \left(\frac{1-t}{1+t} \right), \quad (2.2)$$

with P_N the Legendre polynomial of degree N and \bar{P}_N the associated Legendre polynomial (see the next section for the definition of associated orthogonal polynomials). The factors $(t+1)^N$ in (2.2) cancel out; their role is only to make R_N^0 and S_N^0 polynomials in t . In the Migdal limit, R_N^0 and S_N^0 approach

$$R(t) = -t \log(t) J_0 \left(2\sqrt{\tilde{t}} \right) + \pi t Y_0 \left(2\sqrt{\tilde{t}} \right), \quad (2.3)$$

$$S(t) = J_0 \left(2\sqrt{\tilde{t}} \right). \quad (2.4)$$

On the other hand, a calculation of the two-point boundary correlator for a gauge field in AdS₅ with Neumann boundary conditions at the IR brane yields [5], in the limit $z_{UV} \rightarrow 0$,

$$\Pi(p^2) \sim p^2 \frac{-\log(pz_{UV}) J_0(pz_{IR}) + \pi Y_0(pz_{IR})}{J_0(pz_{IR})} \quad (2.5)$$

We see that identifying $\mu \leftrightarrow z_{UV}^{-1}$, $\tilde{\mu} \leftrightarrow 2z_{IR}^{-1}$ the result $\Pi_{\text{Migdal}} = tR/S$ agrees precisely with the corresponding holographic calculation. Note that, according to the dictionary above and the definition of $\tilde{\mu}$, $2N$ corresponds to the inverse warp factor z_{IR}/z_{UV} .

At this point it is natural to wonder why the Padé approximant chooses Neumann conditions. Our next example shows that we have actually made this choice when identifying the function $f(t)$, leaving the factor t outside of the Padé approximation. Indeed, let us take instead $f(t+1) = -t \log(-t)$, compute the $[N+1/N]$ Padé approximant to $f(t+1)$ at $t = -1$ and define $\Pi_N(t) = [N+1/N]_f = R_N^1/S_N^1$. It is clear that this keeps the same asymptotic function as before. The reason for increasing the degree of the numerator is to improve the convergence at large $|t|$. This technical point will be clarified in the next section. The result is

$$R_N^1(t) = (t+1)^{N+1} \left[-P_N^{(1,0)} \left(\frac{1-t}{1+t} \right) + \bar{P}_N^{(1,0)} \left(\frac{1-t}{1+t} \right) \right], \quad (2.6)$$

$$S_N^1(t) = (t+1)^N P_N^{(1,0)} \left(\frac{1-t}{1+t} \right), \quad (2.7)$$

²We take the principal branch of the logarithm with the branch cut along the real negative axis (argument $\theta \in (-\pi, \pi]$), and define accordingly non-integer powers. The signature of the metric is mostly minus. Physical amplitudes at time-like momenta are evaluated with p^2 right above the positive real axis.

with $(\bar{P}_N^{(\alpha,\beta)}) P_N^{(\alpha,\beta)}$ (associated) Jacobi polynomials. The Migdal limit yields, up to normalization,

$$R(t) = -t \log(t) \frac{1}{\sqrt{\tilde{t}}} J_1 \left(2\sqrt{\tilde{t}} \right) + \pi t Y_1 \frac{1}{\sqrt{\tilde{t}}} \left(2\sqrt{\tilde{t}} \right), \quad (2.8)$$

$$S(t) = \frac{1}{\sqrt{\tilde{t}}} J_1 \left(2\sqrt{\tilde{t}} \right). \quad (2.9)$$

The result $\Pi_{\text{Migdal}} = R/S$ is identical to the holographic one for $z_{\text{UV}} \rightarrow 0$, with the same identifications as before, but this time with Dirichlet boundary conditions at the IR brane.

Now, let us consider a different asymptotic behaviour in the deep Euclidean regime. We assume that the polarization function of two vector currents approaches $-(-t)^{1/2}$ for $t \rightarrow \infty$. This behaviour is very different from the one encountered in QCD. Instead, it is the prediction of a holographic calculation in an asymptotically flat space. Let us first compute $\Pi_N(t) = t[N/N]_f = tR_N^0/S_N^0$ with $f(t+1) = (-t)^{-1/2}$. We obtain

$$R_N^0(t) = (t+1)^N \left[P_N^{-1/2,1/2} \left(\frac{1-t}{1+t} \right) + \bar{P}_N^{-1/2,1/2} \left(\frac{1-t}{1+t} \right) \right], \quad (2.10)$$

$$S_N^0(t) = (t+1)^N P_N^{-1/2,1/2} \left(\frac{1-t}{1+t} \right), \quad (2.11)$$

and in the Migdal limit,

$$R(t) = \frac{1}{\sqrt{t}} \sin \left(2\sqrt{\tilde{t}} \right), \quad (2.12)$$

$$S(t) = \cos \left(2\sqrt{\tilde{t}} \right). \quad (2.13)$$

In this case, $\Pi_{\text{Migdal}} = tR/S$ agrees with the holographic two-point function in 5D Minkowski space with Neumann boundary conditions at the IR brane! Choosing instead $f(t+1) = -(-t)^{1/2}$ and $\Pi_N(t) = [N/N]_f$, we arrive at

$$R_N^0(t) = (t+1)^N \left[-P_N^{1/2,-1/2} \left(\frac{1-t}{1+t} \right) + \bar{P}_N^{1/2,-1/2} \left(\frac{1-t}{1+t} \right) \right], \quad (2.14)$$

$$S_N^0(t) = (t+1)^N P_N^{1/2,-1/2} \left(\frac{1-t}{1+t} \right), \quad (2.15)$$

with the Migdal limit

$$R(t) = -\cos \left(2\sqrt{\tilde{t}} \right), \quad (2.16)$$

$$S(t) = \frac{1}{\sqrt{t}} \sin \left(2\sqrt{\tilde{t}} \right). \quad (2.17)$$

As could be already guessed, $\Pi_{\text{Migdal}} = R/S$ is in this case the same as the holographic two-point function in 5D Minkowski space with Dirichlet boundary conditions on the IR brane.

We have given several examples in which the Migdal procedure in a mysterious way reconstructs the full 5D correlation functions from information about the deep Euclidean

limit. One should be however aware that this “magic” does not always work. For example, let us try to reproduce the result for the two-point function of a scalar operator of conformal dimension 3, which is dual to a massive scalar in AdS₅, with mass $M^2 = -3k^2$. The asymptotic result of the AdS calculation in the deep Euclidean is, up to a constant term, $-t \log(-t)$. This is the same as for gauge bosons, except for the constant term. But the Padé approximant is determined essentially by the non-analytic piece of the function, as we discuss in the next section. Therefore, choosing $f(t+1) = -\log(-t)$ (and multiplying by t at the end) or $f(t+1) = -t \log(-t)$, we arrive at the same Padé approximants as in the first two examples. In the second case, when a local polynomial term is added, we reproduce the holographic result for the massive scalar with Dirichlet boundary conditions. However, the first choice does not give the holographic function for Neumann boundary conditions, but rather one with mixed boundary conditions on the IR brane.

3. Padé approximants and orthogonal polynomials

In this section, following mathematical literature [15], we present a more systematical approach to Padé approximants. Given a function $f(s)$ with a Taylor expansion at $s = 0$ we can define the Padé approximant as follows. We introduce two polynomials $R_N^J(s)$, $S_N^J(s)$ of degrees $N + J$, N , respectively. We choose them such that their ratio has a Taylor expansion at $s = 0$ that matches the Taylor expansion of $f(s)$ up to terms of order s^{2N+J+1} :

$$\frac{R_N^J(s)}{S_N^J(s)} = f(s) + O(s^{2N+J+1}) \tag{3.1}$$

We also assume $S_N^J(0) \neq 0$. The Padé approximant is defined as

$$[N + J/N]_f(s) = \frac{R_N^J(s)}{S_N^J(s)}. \tag{3.2}$$

We will often omit the specification of the function and/or the variable, and write simply $[N + J/N]$. Note that R_N^J and S_N^J are determined up to normalization only, and that both depend on N and J . In the following we will restrict the input function f to be a Stieltjes function. The input functions we studied in the previous section belong to this class up to a certain number of subtractions, as we discuss at the end of this section. A Stieltjes function is defined by the Stieltjes integral representation

$$f(s) = \int_0^{\lambda^{-1}} \frac{d\phi(u)}{1 - su} \tag{3.3}$$

where $\phi(u)$ is a bounded, nondecreasing function on $0 \leq u < \infty$, with finite real-valued moments

$$f_j = \int_0^{\lambda^{-1}} u^j d\phi(u), \quad j = 0, 1, 2, \dots, \tag{3.4}$$

and $\lambda > 0$. An expansion of (3.3) in power series at $s = 0$ gives the Taylor series

$$f(s) = \sum_{j=0}^{\infty} f_j s^j, \tag{3.5}$$

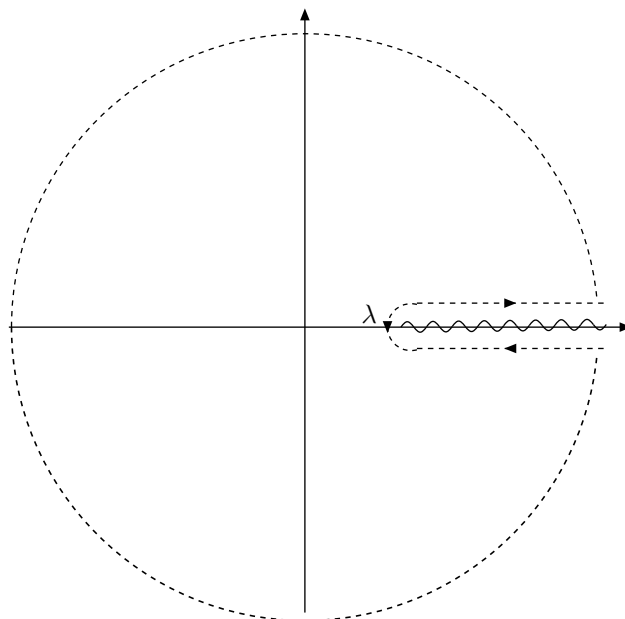


Figure 1: The contour of integration Γ .

which converges in the open disk $|s| < \lambda$. We assume in the following that the function $\phi(u)$ is strictly increasing (and hence the measure strictly positive) except for, at most, a discrete number of points. Then, $f(s)$ has a branch cut along $\lambda < s < \infty$.

There is a remarkable connection between Padé approximants and orthogonal polynomials, which we derive next (for reviews of orthogonal polynomials see, for instance, [16]). From the defining equation (3.1), it is clear that

$$\frac{d^m}{ds^m} (f(s)S_N^J(s))\Big|_{s=0} = 0, \quad m = N + J + 1, \dots, 2N + J. \quad (3.6)$$

Then we can use Cauchy's integral formula to write these conditions in the form of contour integrals:

$$\int_{\Gamma} dz \frac{S_N^J(z)f(z)}{z^{m+1}} = 0, \quad m = N + J + 1, \dots, 2N + J \quad (3.7)$$

with the path Γ displayed in figure 1. The integral along the small semicircle at λ vanishes since, from its definition (3.3), $f(z)$ has at most a logarithmic singularity at $z = \lambda$. The integral along the big circle also vanishes when $J \geq -1$. Finally, the remaining integrals above and below the branch cut cancel except for the discontinuity in the imaginary part of $f(z)$. On the other hand, from (3.3) this jump is related to the measure $d\phi(u) = \phi'(u)du$ by

$$\phi'(u^{-1}) = \frac{u}{2\pi i} (f(u + i\epsilon) - f(u - i\epsilon)), \quad \lambda < u < \infty. \quad (3.8)$$

Therefore, for $J \geq -1$ and any $N \geq 0$,

$$\int_{\lambda}^{\infty} \phi'(u^{-1}) \frac{S_N^J(u)}{u^{m+2}} = 0, \quad m = N + J + 1, \dots, 2N + J. \quad (3.9)$$

Changing variables to $w = u^{-1}$ and shifting $m \rightarrow m - N - J - 1$, we arrive at

$$\int_0^\lambda dw W^J(w) w^m (w^N S_N^J(w^{-1})) = 0, \quad m = 0, \dots, N - 1, \quad (3.10)$$

with $W^J(w) = w^{J+1} \phi'(w)$. The factor in parenthesis is a polynomial of degree N . Eq. (3.10) shows that the set of polynomials

$$\pi_N^J(w) = w^N S_N^J(w^{-1}), \quad N = 0, 1, \dots \quad (3.11)$$

is a system of orthogonal polynomials over the interval $(0, \lambda^{-1})$ with weight $W^J(w)$. We shall also use the notation $d\phi^J(w) = w^{J+1} d\phi(w) = W^J(w) dw$. Conversely,

$$S_N^J(s) = s^N \pi_N(s^{-1}). \quad (3.12)$$

This determines the denominators of the Padé approximant $[N + J/N]$ up to normalization. To calculate the numerators, let us define the function

$$F^J(w) = \int_0^{\lambda^{-1}} \frac{d\phi^J(u)}{w - u} \quad (3.13)$$

and the associated orthogonal polynomials

$$\rho_N^J(w) = \int_0^{\lambda^{-1}} d\phi^J(u) \frac{\pi_N(w) - \pi_N(u)}{w - u}, \quad (3.14)$$

which have degree $N - 1$. Then,

$$\pi_N^J(w) F^J(w) = \rho_N^J(w) + \Delta_N^J(w), \quad (3.15)$$

where

$$\begin{aligned} \Delta(w) &= \int_0^{\lambda^{-1}} d\phi^J(u) \frac{\pi_N(u)}{w - u} \\ &= \frac{1}{w} \int_0^{\lambda^{-1}} d\phi^J(u) \left[1 + \frac{u}{w} + \dots + \left(\frac{u}{w}\right)^{N-1} + \left(\frac{u}{w}\right)^N \left(1 - \frac{u}{w}\right)^{-1} \right] \\ &= w^{-m-1} \int_0^{\lambda^{-1}} d\phi^J(u) \frac{u^N}{1 - u/w} \pi_N \end{aligned} \quad (3.16)$$

is of order w^{-N-1} at large $|w|$. On the other hand, expanding the integrand in the definition of $F(s)$ at $u = 0$ we find

$$F^J(s^{-1}) = s^{-J} \left(f(s) - \sum_{j=0}^J f_j s^j \right). \quad (3.17)$$

Therefore, we obtain

$$f(s) = \sum_{j=0}^J f_j s^j + \frac{s^J \rho_N^J(s^{-1})}{\pi_N^J(s^{-1})} + O(s^{2N+J}), \quad (3.18)$$

so that the numerator of the Padé is

$$R_N^J(s) = S_N^J(s) \sum_{j=0}^J f_j s^j + s^{N+J} \rho_N^J(s^{-1}). \quad (3.19)$$

We see that it is a polynomial of degree $N + J$, as required.

Putting all the pieces together, we have shown that the Padé approximants of a Stieltjes function can be expressed as

$$[N + J/N]_f(s) = \sum_{j=0}^J f_j s^j + \frac{s^J \rho_N^J(s^{-1})}{\pi_N^J(s^{-1})}. \quad (3.20)$$

An important consequence of their relation to orthogonal polynomials is that the Padé approximants (with $J \geq -1$) of Stieltjes functions have only simple poles inside the open interval (λ, ∞) , and all the residues are negative. This is the basic property which allows to make contact with large- N_c and with holography at the classical level. It also follows from (3.16) that the series of Padé approximants $[N + J/N]$ of $f(s)$ converge to $f(s)$ in the limit $N \rightarrow \infty$, for all $J \geq -1$, in the region $|s| < \lambda$. (This is not necessarily so for the Padé approximants of a general function.) The rate of convergence is geometric, as $\Delta(w) \sim O(w^{-N-1})$. This can be extended, with weaker rate of convergence, to the domain $\mathbb{C} \setminus [\lambda, \infty)$.

The orthogonal polynomials on the real line obey a three-term recurrence relation of the form

$$\pi_{N+1}(w) = (a_N w + b_N) \pi_N - c_N \pi_{N-1}, \quad (3.21)$$

with $a_N > 0$, b_N real constants and $c_N = (a_N h_N)/(a_{N-1} h_{N-1})$, where

$$h_N = \langle \pi_N | \pi_N \rangle = \frac{\int dw W(w) \pi_N(w)^2}{\int dw W(w)} \quad (3.22)$$

is the squared norm. We shall call $\bar{a}_N, \bar{b}_N, \bar{c}_N = \bar{a}_N/\bar{a}_{N-1}$ the coefficients with normalization $h_N = 1$. The initial conditions for the recurrence relation are $\pi_{-1} = 0$ (or $c_0 = 0$) and $\pi_0 = C$ a nonvanishing constant. From their definition, it is clear that the associated polynomials ρ_N satisfy the same recurrence relation but with initial conditions $\rho_0 = 0$ and $\rho_1 = a_0 C \int dw W(w)$. From (3.21), a three-term recurrence relation for Padé numerators and denominators (3.23) follows:

$$T_{N+1}^J(s) = (a_N^J + b_N^J s) T_N^J(s) - c_N^J s^2 T_{N-1}^J(s), \quad (3.23)$$

where $T = R, S$ and we have explicitly indicated the dependence of the coefficients on J . The equation is identical for S and R , but the initial conditions are not. These follow from the ones for the orthogonal polynomials and associated orthogonal polynomials, respectively. For instance, when $J = -1$, $R_0^{-1} = 0$, $R_1^{-1} = \text{const}$, $S_{-1}^{-1} = 0$, $S_0^{-1} = \text{const}$. Note the factor s^2 in the last term, which makes the equation different from the one for orthogonal polynomials. It can be shown that, because the weight function W has a compact support, the combinations $1/\bar{a}_N^J$ and \bar{b}_N^J/\bar{a}_N^J will be bounded. It is possible to reverse our

line of argument taking us from a Stieltjes function to the recurrence relation (3.23) for its Padé numerators and denominators. According to Favard's theorem, a series of coefficients $\{\bar{a}_n, \bar{b}_n\}$ with bounded $1/\bar{a}_N^J$ and \bar{b}_N^J/\bar{a}_N^J determines a unique compactly supported measure $d\phi(u)$, and a system of orthogonal polynomials with respect to this measure, such that these coefficients appear in their recurrence relation. The measure, in turn, determines the Stieltjes function.

All the two-point asymptotic functions $f(s)$ which appear in the conformal approximation (which in QCD applies to the leading and subleading perturbative contributions) are of the form $f_n(s) = -(s - \lambda)^n \log(\lambda - s)$, with n a positive integer or zero, or $f_\nu(s) = (-1)^{[\nu]+1}(\lambda - s)^\nu$ with ν a positive non-integer real number and $[\nu]$ the integer closest to ν with $[\nu] \leq \nu$ (the entire part for positive ν). The variable s is related to $t = p^2/\mu^2$ by $s = t + \lambda$. These functions are analytic at $s = 0$, and their Maclaurin series have radius of convergence λ . On the other hand, the divergent behaviour at large $|s|$ can be taken care of by performing $n + 1$ or $[\nu] + 1$ subtractions, respectively. The subtracted functions are Stieltjes functions, and hence their Padé approximants satisfy all the properties we have just derived. For instance, for $f_1(s) = (1 - s) \log(1 - s)$, we need two subtractions:

$$\begin{aligned} \bar{f}_1(s) &= f_1(s) - f_1(0) - s f_1'(0) \\ &= (1 - s) \log(s - 1) + s \\ &= s^2 \int_0^1 \frac{1 - u}{1 - su} du. \end{aligned} \tag{3.24}$$

So, \bar{f}_1/s^2 is a Stieltjes function, analytic inside the open circle of radius 1 centered at $s = 0$. This equation is nothing but (3.17) with $\bar{f}_1(s) = F^J(s^{-1})$ and $J = 2$. Therefore, the function

$$\Pi_N(s) = f_1(0) + s f_1'(0) + s^2 [N - 1/N]_{\bar{f}_1}(s), \tag{3.25}$$

is exactly the same as the Padé approximant $[N + 1/N]_{f_1}(s)$. It is clear that this is generalized to $J = m - 1$ in the case of m subtractions.

Consider any $f_\nu(s)$ with real ν (possibly integer) and compute the $[N + J/N]$ as above, with $J \geq [\nu] - 1$. Changing variables to $x = 2w - 1 = (1 - t)/(1 + t)$ we find a weight $W^J(x) = (1 - x)^\nu (1 + x)^{J-\nu}$ with $x \in (-1, 1)$. Therefore, $\pi_N^J(x) = P_N^{(\nu, J-\nu)}$ and $\rho_N^J(x) = \bar{P}_N^{(\nu, J-\nu)}$, and the Padé approximant is

$$[N + J/N]_{f_\nu} = \sum_{j=0}^J (f_\nu)_j s^j + \frac{s^J \bar{P}_N^{(\nu, J-\nu)} \left(\frac{2-s}{s}\right)}{P_N^{(\nu, J-\nu)} \left(\frac{2-s}{s}\right)}. \tag{3.26}$$

From their integral representation given by (3.14), the associated Jacobi polynomials can be written in terms of hypergeometric functions. Note that we should only consider functions f_ν with $\nu > -1$. In fact, when $\nu \leq -1$, the function f_ν is too divergent at $s = \lambda$ to be a Stieltjes function. Of course, the Padé approximant can still be calculated, but it will not share the good (physical) properties we have derived in this section.

4. Migdal's limit

Let us study the Migdal limit of the Padé recurrence relation (3.23). We take $\lambda = 1$ and call $\tau = \sqrt{t}$ (with τ positive for positive t). the Migdal limit is $N \rightarrow \infty$ and $\tau \rightarrow 0$ with fixed J and $\tilde{\tau} = \tau N$. To find the limit of equation (3.23), we write (suppressing the index J) $T_N(s) = T(N, \tau)$, treat N as a continuous variable and expand

$$T_{N\pm 1}(N, \tau) = T(N, \tau) \pm \frac{\partial}{\partial N} T(N, \tau) + \frac{1}{2} \frac{\partial^2}{\partial N^2} T(N, \tau) + \dots \quad (4.1)$$

Then, keeping only terms up to two derivatives, we get a differential equation of the form

$$\left[\frac{\partial^2}{\partial N^2} + 2 \frac{1 - c_N(1 + \tau^2)^2}{1 + c_N(1 + \tau^2)^2} \frac{\partial}{\partial N} + 2 \frac{1 - a_N - b_N(1 + \tau^2) + c_N(1 + \tau^2)^2}{1 + c_N(1 + \tau^2)^2} \right] T(N, \tau) = 0. \quad (4.2)$$

Let us assume now that the coefficients of the recurrence relation can be expanded at large N as

$$a_N = a^{(0)} + a^{(1)} \frac{1}{N} + a^{(2)} \frac{1}{N^2} + \dots, \quad (4.3)$$

and similarly for b_N and c_N . Then, the second order differential equation (4.2) has a finite non-trivial Migdal limit if and only if the following conditions are met:

$$\begin{aligned} c^{(0)} &= 1, \\ 2 - a^{(0)} - b^{(0)} &= 0, \\ c^{(1)} - a^{(1)} - b^{(1)} &= 0. \end{aligned} \quad (4.4)$$

As long the norm of the orthogonal polynomials can also be expanded at large N (with a finite number of terms with positive powers of N), the first condition is satisfied. The second condition is ensured by Rakhmanov's theorem [18]: if the measure $d\phi$ is supported in $[-1, 1]$ and $\phi' > 0$ almost everywhere in $[-1, 1]$, then it belongs to the Nevai class³ with $\bar{a}^{(0)} = 2$ and $\bar{b}^{(0)} = 0$. The assumptions of the theorem are fulfilled by the measure of Stieltjes functions, in the variable $x = 2\lambda^{-1}w - 1$. Even though the coefficients of the recurrence relation in this variable are different, when $\lambda = 1$ (for which (4.4) apply), the value of $a_n + b_n$ is unchanged. On the other hand, $a^{(0)} = \bar{a}^{(0)}$, $b^{(0)} = \bar{b}^{(0)}$ for the class of norms just mentioned. The third condition is more restrictive. Even if it is norm dependent, it cannot be adjusted without spoiling our assumption that the coefficients can be expanded in $1/N$. We check explicitly below that this condition is fulfilled by Jacobi polynomials.

If all three conditions are met, we find for $N \rightarrow \infty$ the following second-order differential equation:

$$\left[\frac{d^2}{d\tilde{\tau}^2} - c^{(1)} \frac{1}{\tilde{\tau}} \frac{d}{d\tilde{\tau}} + (2 - b^{(0)}) + (c^{(2)} - a^{(2)} - b^{(2)}) \frac{1}{\tilde{\tau}^2} \right] T(\tilde{\tau}) = 0. \quad (4.5)$$

³Measures in the Nevai class are those with finite limits $\bar{a}_n \rightarrow \bar{a}^{(0)}$, $\bar{b}_n \rightarrow \bar{b}^{(0)}$ [17].

In the case of standardized Jacobi polynomials $P_n^{(\alpha,\beta)}$ with any α and β we have, going back to the variable w , $a^{(0)} = 4$, $a^{(1)} = -2$, $b^{(0)} = -2$, $b^{(1)} = 1$, $c^{(0)} = 1$ and $c^{(1)} = -1$ so we see explicitly that all the conditions (4.4) are directly satisfied. In this case, (4.5) reads

$$\left[\frac{d^2}{d\tilde{\tau}^2} + \frac{1}{\tilde{\tau}} \frac{d}{d\tilde{\tau}} + 4 - \frac{\alpha^2}{\tilde{\tau}^2} \right] T(\tilde{\tau}) = 0. \tag{4.6}$$

This is a Bessel equation, with general solution

$$T(\tilde{\tau}) = C_1 J_\alpha(2\tilde{\tau}) + C_2 Y_\alpha(2\tilde{\tau}). \tag{4.7}$$

Actually, in the more general case of eq. (4.5) we can write $T(\tilde{\tau}) = \tilde{\tau}^{(1+c^{(1)})/2} V(\tilde{\tau})$ and rescale the variable to $\tilde{\sigma} = \left(\sqrt{2 - b^{(0)}}/2 \right) \tilde{\tau}$. Then, $V(\tilde{\sigma})$ obeys (4.6) with $\tilde{\tau} \rightarrow \tilde{\sigma}$ and $\alpha^2 = a^{(2)} + b^{(2)} - c^{(2)} + (1 + c^{(1)})^2/4$. Note that the common factor $\tilde{\tau}^{(1+c^{(1)})/2}$ will cancel out in the quotient R/S . Actually, this factor comes from the normalization of the orthogonal polynomials. On the other hand, the rescaling of $\tilde{\tau}$ amounts to a rescaling of $\tilde{\mu}$. Therefore, we see that, as far as the Migdal limit is concerned, and if the limit exists, it is sufficient to consider the limit of Jacobi polynomials and work with equation (4.6).

When the third condition is not fulfilled, the recurrence relation does not have a good Migdal limit. In these cases, one could still try to find a continuous differential equation by modifying the way in which the limit is taken, and this was actually done (in a different language) in Migdal's original paper [8]. In the following we consider only the simplest case in which the Migdal limit, as defined here, is finite. We have seen that this reduces to studying the Padé approximants of "conformal" functions f_n and f_ν .

5. The AdS/Migdal correspondence

In this section, we describe the general relation between the Migdal approximation and extra dimensions, for two-point functions. We give a simple argument showing that the Migdal limit unavoidably gives a result which corresponds to a holographic calculation in certain 5D geometries. The converse result does not always hold: in some cases the holographic results cannot be obtained from a Migdal approximation to their asymptotic Euclidean functions. In order to simplify the notation we consider correlators of scalar operators, and comment at the end on the generalization to higher spins.

We start with the field theory (Migdal) side, and show that analyticity⁴ and the condition of finite Migdal limit, together with information about the deep Euclidean limit and the leading infrared behaviour, completely fix the two-point function $\Pi_{\text{Migdal}}(t)$. We have seen at the end of the previous section that a good Migdal limit necessarily gives numerators R and denominators S which satisfy the differential equation (4.6). Therefore, R and S have the form (4.7), with coefficients $C_{1,2}$ which are functions of τ .⁵

⁴The numerators and denominators are analytic in s since they are finite limits of polynomials, and hence convergent Taylor series at $s = 0$.

⁵We are treating $\tau = p/\mu$ and $\tilde{\tau} = p/\tilde{\mu}$ as independent variables. In the following we also need to use the behaviour in the variable p^2 , for fixed $\mu, \tilde{\mu}$.

Now, let us impose the asymptotic value of the two-point function. The limit $|\tilde{\tau}| \rightarrow \infty$ is equivalent to $N \rightarrow \infty$ with fixed τ . Hence, for any non-positive τ^2 , the Padé approximant converges to $f(\tau^2 + 1)$. Therefore, the Migdal limit of the Padé approximant must have the form

$$\frac{R}{S} = \frac{f(\tau^2 + 1)J_\mu(2\tilde{\tau}) + A(\tau)H_\mu^{(1)}(2\tilde{\tau})}{J_\mu(2\tilde{\tau}) + B(\tau)H_\mu^{(1)}(2\tilde{\tau})}, \quad (5.1)$$

with $H_\mu^{(1)}$ the first Hankel function, which goes to zero exponentially for large positive imaginary part of the argument.

Next, we impose that R and S be analytic in p^2 at $p^2 = 0$. This fixes the functions $A(\tau)$ and $B(\tau)$. Recall that for integer n , $J_n(z)$ is an entire function and $Y_n(z)$ equals $2/\pi \log z$ plus an entire function, whereas for any μ , $z^{-\mu}J_\mu(z)$ is entire. Consider first $f_n(s) = -\tau^{2n} \log(-\tau^2)$. Then, the index of the Bessel functions must be an integer, and

$$\frac{R}{S} = \frac{-\tau^{n-m} \log(\tau^2)J_{n+m}(2\tilde{\tau}) + \pi\tau^{n-m}Y_{n+m}(2\tilde{\tau})}{\tau^{-n-m}J_{n+m}(2\tilde{\tau})}. \quad (5.2)$$

On the other hand, for $f_\nu(s) = (-1)^{[\nu]+1}(-\tau^2)^\nu$ and ν a non-integer real, analyticity requires $\mu = \nu + m$ with integer m , and

$$\frac{R}{S} = \frac{(-1)^{[\nu]+1}\tau^{\nu-m}J_{-\nu-m}(2\tilde{\tau})}{\tau^{-\nu-m}J_{\nu+m}(2\tilde{\tau})}. \quad (5.3)$$

Finally, we use the fact that the Padé approximants have no poles or zeros at $p^2 = 0$, and assume that this still holds in the Migdal limit.⁶ The leading behaviour at small p^2 is $R/S \sim p^{-2m}$ in all cases. Hence, we see that $m = 0$ and this gives the final result for R/S . Now, let us define the Migdal two-point function as $\Pi_N^{J,l} = t^{-l}[N + J/N]_{t^l}^f$, with integer l . As discussed above, we should only consider $l > -1 - \nu$. Since R/S has no poles or zeros at $p^2 = 0$, Π_{Migdal}^l will have a zero of degree $-l$ if $l < 0$, and a pole of degree l if $l > 0$. On the other hand, the asymptotic behaviour is independent of l . So, we can use the result above with $m = l$ and find

$$\Pi_{\text{Migdal}}^l = \frac{-t^n \log(t)J_{n+l}(2\sqrt{\tilde{t}}) + \pi t^n Y_{n+l}(2\sqrt{\tilde{t}})}{J_{n+l}(2\sqrt{\tilde{t}})}, \quad (5.4)$$

$$\Pi_{\text{Migdal}}^l = \frac{(-1)^{[\nu]+1}t^\nu J_{-\nu-l}(2\sqrt{\tilde{t}})}{J_{\nu+l}(2\sqrt{\tilde{t}})}, \quad (5.5)$$

in the integer and non-integer cases, respectively. Conversely, we see that we can reproduce a low-energy behaviour $\sim p^{-2l}$ in the two-point function simply by choosing l in the definition of the Migdal two-point function.

We turn now to the holographic calculations in 4+1 dimensions. In order to keep 4D Poincaré invariance, the geometry must be a warped direct product of Minkowski times a one dimensional space I , which can be chosen flat. In order to have a discrete spectrum,

⁶This may be proven using the asymptotic distribution of zeros of the orthogonal polynomials.

I must be compact. As long as the warp factor is strictly monotonic, one can define coordinates in which the metric is manifestly conformally flat:

$$ds^2 = \xi(z)^2 (dx^\mu dx_\mu - dz^2). \tag{5.6}$$

In the coordinate z , $I = [z_{\text{UV}}, z_{\text{IR}}]$. AdS geometry corresponds to $\xi(z) = (kz)^{-1}$; in this case, the UV (IR) boundaries at z_{UV} (z_{IR}) hide the AdS boundary (horizon) at $z = 0$ ($z = \infty$). The holographic prescription to calculate correlation functions of field-theory operators at large N (and strong t'Hooft coupling) is given by the AdS/CFT correspondence [2]: calculate the value of the action for on-shell bulk fields with fixed UV boundary values, which act as sources for the dual operators; then, differentiate functionally with respect to the sources and put them to zero. For two-point functions, the on-shell action reduces to a boundary term. For scalars,

$$\Pi(p) = \lim_{z_{\text{UV}} \rightarrow 0} \left\{ X \left[\frac{\partial_z G(z, p)}{G(z, p)} \right]_{z \rightarrow z_{\text{UV}}} + \text{counterterms} \right\}, \tag{5.7}$$

where $G(z, p)$ is the bulk-to-boundary propagator in a mixed position-momentum representation, fulfilling some specified boundary conditions on the IR boundary and free on the UV. X is a p -independent factor which cancels a divergent factor in the non-analytic part in the limit $z_{\text{UV}} \rightarrow 0$. The remaining counterterms in (5.7) form a polynomial in p^2 which cancels the poles at $z_{\text{UV}} = 0$. The propagator G satisfies the equation of motion for the dual field. We can write a generic IR boundary condition as $\hat{G}(z_{\text{IR}}, p) = 0$, with $\hat{f}(z, p) = \kappa_1(p^2 z^2) f(z, p) + \kappa_2(p^2 z^2) z \partial_z f(z, p)$. Such a boundary condition can be obtained including mass, kinetic and higher-derivative terms localized on the IR boundary. Indeed, higher derivatives of G in the variable z can be written in terms of G and $\partial_z G$ using the bulk equation of motion. Let J and Y be two independent solutions of the equation of motion. Then, we can write

$$G(z_{\text{IR}}, p) = \hat{J}(z_{\text{IR}}, p) Y(z, p) - \hat{Y}(z_{\text{IR}}, p) J(z, p). \tag{5.8}$$

Taking the limit, the holographic formula in which the IR conditions are manifest has the form

$$\Pi(p) = \frac{A_1(p) \hat{J}(z_{\text{IR}}, p) - A_2(p) \hat{Y}(z_{\text{IR}}, p)}{A_3(p) \hat{J}(z_{\text{IR}}, p) - A_4(p) \hat{Y}(z_{\text{IR}}, p)} + \text{local terms}. \tag{5.9}$$

Our aim is to relate the differential equation satisfied by the limit of the Padé numerator and denominator to a differential equation for $\hat{K}(z_{\text{IR}}, p)$ in the variable z_{IR} , where K is any linear combination of J and Y . Then, meromorphicity, which follows from the discreteness of the spectrum in a compact space, will imply that the holographic two-point function will be the same as the Migdal one for some value of l . The function f in the Migdal approach corresponds to the limit $z_{\text{IR}} \rightarrow \infty$ of (5.7), with $p^2 < 0$. In fact, for the functions we are considering, which only depend on p^2/μ^2 , we should take as well a low-energy limit in which all the scales but $|p|$ (and $1/z_{\text{IR}}$, which we have sent to zero) go to infinity. Then, the equivalence of the complete function will hold only in this limit (with finite z_{IR}).

In terms of the length variable $L = 2/\tilde{\mu}$, eq. (4.6) reads

$$\left[\frac{d^2}{dL^2} + \frac{1}{L} \frac{d}{dL} + p^2 - \frac{\alpha^2}{L^2} \right] T(Lp/2) = 0. \quad (5.10)$$

Note that dimensional analysis plus the fact that p^2 appears only as $p^2 T$ completely fix the form of this equation. On the other hand, the equation of motion of a scalar of mass M is

$$\left[\partial_z^2 + 3(\partial_z \log \xi(z)) \partial_z + p^2 - \xi^2(z) M^2 \right] \phi(z, p) = 0. \quad (5.11)$$

In the case of Dirichlet boundary conditions, $\kappa_1 = 1$, $\kappa_2 = 0$, and $\hat{K}(z_{\text{IR}}, p) = K(z_{\text{IR}}, p)$ obeys the same equation as $\phi(z, p)$:

$$\left[\partial_{z_{\text{IR}}}^2 + 3(\partial_{z_{\text{IR}}} \log \xi(z_{\text{IR}})) \partial_{z_{\text{IR}}} + p^2 - \xi(z_{\text{IR}})^2 M^2 \right] \hat{K}(z_{\text{IR}}, p) = 0. \quad (5.12)$$

We see that, because we are using conformal coordinates, the normalization of the term with p^2 in (5.12) is the same as in (5.10). If $M \neq 0$, complete agreement with the Migdal equation (5.10) requires that $\xi(z) = (kz)^{-1}$, with k a constant with dimensions of mass. Therefore, the space must be a slice of AdS with curvature k . In this case, $3\partial_{z_{\text{IR}}} \log \xi = -3/z_{\text{IR}}$. To go to the normalization of (5.10) we only need to write $\hat{K}(z_{\text{IR}}, p) = z_{\text{IR}}^2 \hat{H}(z_{\text{IR}}, p)$. Then, for $z_{\text{IR}} = L$ and $m^2 \equiv M^2/k^2 = \alpha^2 - 4$, the equation for \hat{H} is exactly the same as eq. (5.10). From the AdS/CFT relation between the conformal dimension of the operator Δ and the mass of the dual field, we see that $\Delta = \alpha + 2$, as it should (remember that α is the exponent of the asymptotic two-point function, which is determined by conformal invariance). On the other hand, in the case $M = 0$, we find agreement with Migdal equation for any $\xi(z) = (kz)^\eta$. Then, we define $\hat{K}(z_{\text{IR}}, p) = z_{\text{IR}}^{(1-3\eta)/2} \hat{H}(z_{\text{IR}}, p)$ and identify $\alpha^2 = (3\eta - 1)^2/4$. Note that when $\alpha \neq 0$ there are two different values of η which give the same α . Of course, for $\eta \neq -1$ the geometry is not asymptotically AdS, and the AdS/CFT dictionary linking masses to conformal dimensions must be modified. In the particular case $\eta = 0$ we have flat space, and we see that the same Dirichlet two-point function, behaving asymptotically like $-(-t)^{1/2}$, is found in flat space with a massless scalar and in AdS with $m^2 = -15/4$. At any rate, adjusting the mass parameter m we can always reproduce Migdal's differential equation in AdS with Dirichlet boundary conditions in the IR. Moreover, Dirichlet boundary conditions give a correlator which (for scalars) has not a zero nor a pole at $p^2 = 0$. Indeed, we can rescale the field with a z -dependent factor such that the mass term in the equation of motion cancels. This shows that any zero-mode must be flat. But then, the Dirichlet condition forces it to vanish. This is true for any UV boundary condition. For UV Dirichlet (Neumann) conditions, this is telling us that the (inverse) two-point function does not have a pole at zero momentum.⁷ Therefore, we conclude that IR Dirichlet boundary conditions correspond to $l = 0$ in the Migdal approach.

Consider now Neumann boundary conditions, $\hat{K} = \partial_z K$. As long as $M = 0$, and for any η , \hat{K} satisfies the following second-order differential equation:

$$\left[\partial_{z_{\text{IR}}}^2 + \frac{3\eta}{z_{\text{IR}}} \partial_{z_{\text{IR}}} + p^2 - \frac{3\eta}{z_{\text{IR}}^2} \right] \hat{K}(z_{\text{IR}}, p) = 0. \quad (5.13)$$

⁷For the relation between holographic correlators and connected correlation functions of the AdS theory, including the propagator, see the third reference in [5].

Writing $\hat{K} = z_{\text{IR}}^{(1-3\eta)/2} \hat{H}$, we reproduce Migdal equation (5.10) for $\alpha^2 = (3\eta + 1)^2/4$. Thus, $\alpha_{\text{Dirichlet}}^2 - \alpha_{\text{Neumann}}^2 = -3\eta$. For positive α and $\eta \leq -1/3$, $\alpha_{\text{Dirichlet}} - \alpha_{\text{Neumann}} = 1$. Furthermore, in this case there would be a zero mode if Neumann boundary conditions were also used in the UV, so the two-point function has a simple zero. Therefore, IR Neumann boundary conditions in a massless theory corresponds to choosing $l = -1$ in the Migdal approach. On the other hand, if $M \neq 0$, it is possible to write a second-order differential equation for \hat{K} , but with coefficients which are not analytic in p^2 . So, this equation is not of the Migdal form and we cannot reproduce the holographic function within the Migdal approach.

Let us study next mixed boundary conditions with $\kappa_1 = 1$ and κ_2 a constant, and assume $\eta = -1$ to start with. Then, the function \hat{K} satisfies a differential equation of the Migdal form if and only if $\kappa_2 = \frac{2 \pm \sqrt{4+m^2}}{m^2}$. After rescaling, we find $\alpha^2 = 5 + m^2 \mp 2\sqrt{4 + m^2}$, respectively. It turns out that this boundary condition, which can be understood as arising from a mass term localized on the IR brane, is automatic when the scalar field is the supersymmetric partner of a fermion or a gauge boson [19]. In this case, a fine-tuned mixed UV boundary condition arises as well, such that a zero-mode results. Therefore, this corresponds again, up to local terms, to a Migdal function with $l = -1$. On the other hand, if $M = 0$ and η arbitrary, $\kappa_2 = 1/(3\eta - 1)$ and $\alpha^2 = [3(\eta - 1)/2]^2$. Note that the remaining solution which would correspond to infinite κ_2 , is the one studied in the Neumann case.

It is also possible to reproduce any integer value of l with $l + \alpha_{\text{Dirichlet}} > -1$ by choosing adequate analytic functions κ_1 and κ_2 . This can be proven showing that a good differential equation for \hat{K} is obtained for discrete values of the coefficients (depending on the mass) in the expansions of κ_1 and κ_2 , and studying their behaviour at small momentum. Describing this in detail would be lengthy, so we simply observe that these properties follow quite straightforwardly from the differential-recursion relations of Bessel functions and leave the details to the interested reader.

So, to summarize, every Migdal approximation of a two-point function in the (finite) Migdal limit can be reproduced by an AdS calculation with a given mass and fine-tuned boundary conditions. The Euclidean asymptotic behaviour is determined by the mass, whereas the different discrete choices of the parameter l , which controls the leading behaviour at $p^2 = 0$, correspond to different discrete IR boundary conditions in AdS. The converse is not true: not for any mass and IR boundary condition can one find an equivalent Migdal approximation. One can alternatively reproduce the Migdal calculations using a different warped geometry when the mass of the dual field vanishes. In all cases, the infrared Migdal parameter $\tilde{\mu}$ is proportional to the inverse of the position of the IR boundary in the conformal coordinates (the ones for which the metric is manifestly conformally flat).

All this discussion can be readily extended to higher integer spins. Ultraviolet conformal invariance determines the form of the two-point function in the UV as

$$\langle O_i O_j \rangle = Z_{ij}(p)/p^{2n} f(p^2 + 1), \tag{5.14}$$

where i, j represent Lorentz indices and Z_{ij} is a tensor, polynomial in p_μ . We choose n such that Z_{ij}/p^{2n} be adimensional. For instance, for a vector operator of conformal

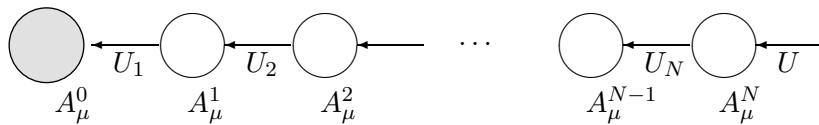


Figure 2: The moose diagram for our deconstruction setup

dimension Δ ,

$$Z_{\mu\nu} = \left(\frac{2(\Delta - 2)}{\Delta - 1} p_\mu p_\nu - \eta_{\mu\nu} p^2 \right) \quad (5.15)$$

and $n = 1$. For a conserved current, $\Delta = 3$ and we get a transverse function. We can directly apply Migdal's approximation to f , and define

$$\Pi_{\text{Migdal } ij}^l = Z_{ij}(p)/p^{2n} \Pi_{\text{Migdal}}(p^2)^l. \quad (5.16)$$

This keeps the tensorial form dictated by conformal invariance. On the other hand, the holographic calculation will preserve the conformal tensor structure if the geometry is that of a slice of AdS. It will also preserve this form for any metric in the case of completely antisymmetric tensors which are dual to p -forms, due to gauge invariance. Therefore, the results we have obtained for massless scalars can be generalized to higher p -forms, and in particular to gauge fields. For instance, the unphysical example in section 2 with asymptotic behaviour $\Pi(p^2) \sim p$ corresponds to a gauge field in flat space. An AdS calculation with a vector field with $m^2 = -3/4$ (corresponding to conformal dimension $5/2$) and adequate boundary terms would reproduce the scalar function $\Pi(p^2)$, but not a transverse tensor. Finally, in extending the discussion for scalars to tensors, one should also take into account that the coefficients in the second term of the equation of motion (5.11) will be different, which leads to a different normalization and to a different relation between m and α (or, equivalently, between m and Δ). For vector bosons, the coefficient 3 is changed to 1 and $\alpha_{\text{Dirichlet}} = \sqrt{1 + m^2}$.

6. Deconstruction and holography

In the previous section we have studied the Migdal limit of Padé approximants. In this one, we relate the approximants with finite N to deconstruction models. In particular, this makes explicit the mechanism by which Migdal correlators approach the holographic ones, and how a discrete version of the holographic formula (5.7) is realized by the Padé approximants. For definiteness, we will stick to the case of gauge bosons — dual to conserved currents — and discuss Dirichlet and Neumann conditions only.

We consider a deconstruction model corresponding to the moose diagram sketched in figure 2. It involves a chain of N $SU(N_F)$ groups with the gauge fields $A_\mu^j = A_\mu^{j,a} T^a$, $j = 1 \dots N$. The groups communicate with nearest neighbours via bifundamental non-linear sigma model field U_j , referred to as the links. They are unitary $N_F \times N_F$ matrices with determinant equal to 1. The role of the last non-linear sigma field U is to control the boundary conditions for the gauge field (the analogue of IR boundary conditions in 5D).

At the left end of the chain we singled out the “boundary” gauge field A_μ^0 . Similarly as in AdS/CFT, this boundary field is interpreted as an external current probing the dynamics of the “bulk” model. Here, the bulk refers to the remaining gauge fields A_μ^j , $j \geq 1$. The latter will show up as resonances in the boundary field correlation function.

In the following we calculate the two-point correlation function of the boundary fields in deconstruction. We do it first in the more familiar minimal deconstruction set-up, and then in what we call tilted deconstruction, which contains additional interactions between the neighbouring sites. Correlators obtained in tilted deconstruction turn out to be directly related to Migdal’s Padé approximants and we work out a dictionary between the two approaches.

6.1 Minimal deconstruction

The gauge transformations ω_j , $j = 0 \dots N$, act as $A_\mu^j \rightarrow \omega_j A_\mu \omega_j^\dagger - i \partial_\mu \omega_j \omega_j^\dagger$ and $U_j \rightarrow \omega_{j-1} U_j \omega_j^\dagger$. The simplest non-trivial action that is invariant under these transformations can be written as

$$S = \int d^4x \sum_{j=1}^N \left(-\frac{1}{2g_j^2} \text{tr}\{F_{\mu\nu}^j F_{\mu\nu}^j\} + \text{tr}\{v_j^2 D_\mu U_j D_\mu U_j^\dagger\} \right) + \int d^4x \text{tr}\{v^2 D_\mu U D_\mu U^\dagger\}, \quad (6.1)$$

with $D_\mu U_j = \partial_\mu U_j - i A_\mu^{j-1} U_j + i U_j A_\mu^j$, $D_\mu U = \partial_\mu U - i A_\mu^N U$. This action is minimal in the sense that the only interactions between various gauge fields come from the covariant derivatives acting on the links. The relation of this deconstruction setup to 5D gauge theories can be worked out analogously as in [14]. Consider the 5D action for a gauge field in the background $ds^2 = a^2(z)dx^2 - b^2(z)dz^2$:

$$\begin{aligned} S_{5D} &= \int d^4x \int_0^L dz \sqrt{g} \left(-\frac{1}{2g_5^2} \text{tr} F_{MN}^2 \right) \\ &\rightarrow \int d^4x \int_0^L dz \left(-\frac{b(z)}{2g_5^2} \text{tr} F_{\mu\nu}^2 + \frac{a^2(z)}{g_5^2 b(z)} \text{tr} (\partial_5 A_\mu)^2 \right). \end{aligned} \quad (6.2)$$

Latticizing the 5th coordinate, $z \rightarrow z_j = j\Delta$, $\partial_5 f(z) \rightarrow (f(z_j) - f(z_{j-1}))/\Delta$ we obtain:

$$S_{5D} \rightarrow \int d^4x \sum_{j=1}^N \left(-\frac{b(z_j)\Delta}{2g_5^2} \text{tr} F_{\mu\nu}^2(z_j) + \frac{a^2(z_j)}{g_5^2 \Delta b(z_j)} \text{tr} (A_\mu(z_j) - A_\mu(z_{j-1}))^2 \right). \quad (6.3)$$

This can be mapped onto the deconstruction action (6.1) (in the unitary gauge $U_j = 1$). The warp factors and the lattice spacing translate into the parameters of the deconstruction lagrangian according to the following dictionary:

$$a(z_j) \rightarrow \frac{v_j g_1}{v_1 g_j}, \quad b(z_j) \rightarrow \frac{g_1^2}{g_j^2}, \quad \Delta \rightarrow \frac{1}{g_1 v_1}, \quad g_5^2 \rightarrow \frac{g_1}{v_1}. \quad (6.4)$$

We have fixed $a(z_1) = b(z_1) = 1$. In passing we note that discretization in Poincaré coordinates, $b(z) = 1$, corresponds to $g_j = g$, while discretization in conformal coordinates, $b(z) = a(z)$, corresponds to $g_j v_j = gv$.

We can integrate out all the bulk gauge fields and obtain an effective action for the boundary field. At tree-level, the integrating-out amounts to 1) solving the equations of motion for the bulk fields in the presence of a background boundary field and 2) inserting the solution into the deconstruction action. The details of this procedure are given in the appendix. At the quadratic level, the effective action in the momentum space has the form

$$S_{\text{eff}} = \int \frac{d^4 p}{(2\pi)^4} v_1^2 A_\mu^0(p) \left(-\eta_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} \right) A_\nu^0(p) \Pi(p^2). \quad (6.5)$$

The polarization function is given by a compact expression

$$\Pi(p^2) = \frac{F_N^1(p^2)}{F_N^0(p^2)} + \frac{1}{g_0^2 v_1^2} p^2 - 1, \quad (6.6)$$

where F_N^j is a solution to the equation of motion

$$\left(v_{j+1}^2 + v_j^2 - \frac{p^2}{g_j^2} \right) F_N^j - v_j^2 F_N^{j-1} - v_{j+1}^2 F_N^{j+1} = 0, \quad (6.7)$$

subject to a boundary condition at $j = N$. The boundary condition is controlled by the parameter v in the lagrangian. The limit $v \rightarrow 0$ leads to a deconstructed analogue of the Neumann boundary condition,

$$F_N^{N+1} = F_N^N, \quad (6.8)$$

while $v \rightarrow \infty$ corresponds to the Dirichlet boundary condition,

$$F_N^N = 0. \quad (6.9)$$

Intermediate values of v correspond to mixed boundary conditions. On the other hand, for $g_0 \rightarrow \infty$ we recover the case of non-dynamical boundary fields, which can be regarded as sources.

There are many apparent similarities between polarization functions obtained in minimal deconstruction and those derived using the Migdal procedure. Let us point them out.

- i) The correlation function is represented as a ratio of two polynomials in p^2 :

$$\Pi(p^2) = \frac{R_N(p^2)}{S_N(p^2)}, \quad (6.10)$$

where

$$R_N = F_N^1 + F_N^0 \left(\frac{p^2}{g_0^2 v_1^2} - 1 \right) \quad S_N = F_N^0 \quad (6.11)$$

Indeed, from the equation of motion (6.7) F_N^j is a polynomial of degree $N - j$ in p^2 , once we set $F_N^N = \text{const}$ (in the Dirichlet case $F_N^{N-1} = \text{const}$ and F_N^j has the degree $N - j - 1$).

- ii) The numerator and the denominator satisfy a second-order recurrence relation in the degree N , and the equation is the same for both. For example, in the Dirichlet case it is given by ($T = R, S$):

$$T_{N+1}(p^2) = \left(1 + \frac{v_N^2}{v_{N+1}^2} - \frac{p^2}{g_N^2 v_{N+1}^2}\right) T_N(p^2) - \frac{v_N^2}{v_{N+1}^2} T_{N-1}(p^2). \quad (6.12)$$

This follows from the fact that the solution F_N^j satisfying the Dirichlet boundary conditions can be written as $F_N^j = Y_N J_j - J_N Y_j$, where J_j and Y_j are any two independent solutions to eq. (6.7).

- iii) In the limit $N \rightarrow \infty$ and for $|p^2| \ll v_1^2$ and $p^2 < 0$, the polarization function obtained in deconstruction approximates the non-analytic behaviour of the polarization function in the corresponding 5D model in the deep Euclidean regime. For example, in the deconstructed AdS models one obtains [13] $\Pi(p^2) \sim -p^2 \log(-p^2/v_1^2)$, while in the deconstructed flat models we find $\Pi(p^2) \sim p^2(-p^2/v_1^2)^{-1/2}$.

In spite of these similarities it is not possible to find a precise mapping between the Migdal approximation and minimal deconstruction. The reason is that the recurrence relations (6.12) and (3.23) are incompatible. Indeed, the form of the recurrence relation (6.12) implies that $T_N(p^2)$ is an orthogonal polynomial in the variable p^2 . On the other hand, the numerators and denominators obtained by the Migdal procedure, although related to orthogonal polynomials by eq. (3.12), are themselves *not* orthogonal polynomials.

In the following we explore a modified deconstruction framework that allows for a mapping of the boundary correlators to those obtained using the Migdal approximation.

6.2 Tilted deconstruction

We modify the minimal deconstruction action (6.1) adding a kinetic mixing between neighbouring gauge fields,

$$S = \int d^4x \sum_{j=1}^N \left(-\frac{1}{2g_j^2} \text{tr}\{F_{\mu\nu}^j F_{\mu\nu}^j\} + \text{tr}\{v_j^2 D_\mu U_j D_\mu U_j^\dagger\} \right) + \int d^4x \text{tr}\{v^2 D_\mu U_{N+1} D_\mu U_{N+1}^\dagger\} \\ - \int d^4x \sum_{j=1}^N \frac{\alpha_j}{2g_j^2} \text{tr}\{F_{\mu\nu}^{j-1} U_j F_{\mu\nu}^j U_j^\dagger + \text{h.c.}\} + \int d^4x \frac{1}{2g^2} \text{tr}\{F_{\mu\nu}^N F_{\mu\nu}^N\} \quad (6.13)$$

Such deconstruction setup is also related to a latticized 5D gauge theory in the warped background. The difference with the minimal case is that the 5D action must contain a higher derivative term breaking the 5D Lorentz invariance:

$$S_{5D} \rightarrow \int d^4x \int_0^L dz \left(-\frac{b(z)}{2g_5^2} \text{tr} F_{\mu\nu}^2 + \frac{a^2(z)}{g_5^2 b(z)} \text{tr} (\partial_5 A_\mu)^2 - \frac{\alpha(z)}{2g_5^2} \text{tr} F_{\mu\nu} \partial_z^2 F_{\mu\nu} \right). \quad (6.14)$$

The dictionary between deconstruction and 5D is now given by:

$$a(z_j) \rightarrow \frac{v_{j+1}}{v_1} \frac{g_1}{g_j} \frac{\sqrt{1-\alpha_j}}{\sqrt{1-\alpha_1}}, \quad b(z_j) \rightarrow \frac{g_1^2}{g_j^2} \frac{1-\alpha_j}{1-\alpha_1}, \quad \alpha(z) \rightarrow \frac{\alpha_j}{g_j^2 v_1^2}, \\ \Delta \rightarrow \frac{\sqrt{1-\alpha_1}}{g_1 v_1}, \quad g_5^2 \rightarrow \frac{g_1}{v_1 \sqrt{1-\alpha_1}}. \quad (6.15)$$

We see that tilted deconstruction corresponds to a 5D setup with some specific higher-derivative terms. Nevertheless, the extra term is irrelevant at low energies and a standard lowest-order 5D action is recovered. Hence, the effect of the mixing term amounts to a renormalization of the coefficients in this action. Viewed as a 4D field-theoretical model, tilted deconstruction is healthy as long as the mixing coefficients α_j are not too large (so that there are no ghosts).

As in the minimal setup, it makes sense to integrate out the resonances A_μ^j , $j \geq 1$ and calculate the effective action for A_μ^0 . The polarization function is given by (see appendix A for a derivation)

$$\Pi(p^2) = \frac{F_N^1(p^2)}{F_N^0(p^2)} \left(1 + \frac{\alpha_1}{g_1^2 v_1^2} p^2 \right) + \frac{1}{g_0^2 v_0^2} p^2 - 1. \quad (6.16)$$

The last two terms have a form of a local polynomial in p^2 . By adding higher derivative terms for the boundary fields to the deconstruction action we could, in fact, obtain an arbitrary polynomial in p^2 .

In tilted deconstruction, F_N^j solve a modified equation of motion

$$\left(v_{j+1}^2 + v_j^2 - \frac{p^2}{g_j^2} \right) F_N^j - \left(v_j^2 + \frac{p^2 \alpha_j}{g_j^2} \right) F_N^{j-1} - \left(v_{j+1}^2 + \frac{p^2 \alpha_{j+1}}{g_{j+1}^2} \right) F_N^{j+1} = 0, \quad (6.17)$$

subject to the boundary condition

$$\left(v_{N+1}^2 - v^2 + \frac{1}{g^2} p^2 \right) F_N^N = \left(v_{N+1}^2 + \frac{\alpha_{N+1}}{g_{N+1}^2} p^2 \right) F_N^{N+1}. \quad (6.18)$$

In the limit $v \rightarrow \infty$ we obtain Dirichlet boundary conditions, $F_N^N = 0$, while setting $v = 0$, $\alpha_{N+1}/g_{N+1}^2 = 1/g^2$ leads to Neumann boundary conditions, $F_N^{N+1} = F_N^N$.

We will prove that, for certain choices of the coefficients α_j , the polarization functions obtained in this setup are directly related to those obtained by Migdal approximation.

6.3 Migdal-deconstruction map

Let us make the following ansatz for the mixing coefficients:

$$\frac{\alpha_j}{g_j^2} = \frac{v_j^2}{\mu^2}, \quad (6.19)$$

where μ is an arbitrary scale. We also introduce a new variable, $s = 1 + p^2/\mu^2$. The equation of motion now becomes

$$\left(v_{j+1}^2 + v_j^2 + \frac{\mu^2}{g_j^2} - s \frac{\mu^2}{g_j^2} \right) F_N^j(s) - v_j^2 s F_N^{j-1}(s) - v_{j+1}^2 s F_N^{j+1}(s) = 0. \quad (6.20)$$

We find it convenient to discuss the Dirichlet and the Neumann case separately.

Dirichlet boundary conditions. We investigate the solutions to eq. (6.20) subject to the boundary condition $F_N^N = 0$. If we set $F_N^{N-1} = \text{const}$ then $s^j F_N^{N-j-1}(s)$ is a polynomial in s of degree j . We define the polynomials

$$R_N^1(s) = s^{N+1} F_{N+1}^1(s), \quad S_N^1(s) = s^N F_{N+1}^0(s). \quad (6.21)$$

It follows that S_N has degree N , while R_N has degree $N+1$. From eq. (6.16) the polarization function can be written as

$$\Pi(s) = \frac{R_N^1(s)}{S_N^1(s)} + f_0 + f_1 s \quad (6.22)$$

and has the form of a Padé approximant with $J = 1$. Indeed, the numerator and the denominator as defined in eq. (6.21) satisfy the second order difference equation

$$T_{N+1}^1(s) = \left(1 + \frac{v_{N+1}^2}{v_{N+2}^2} + \frac{\mu^2}{g_{N+1}^2 v_{N+2}^2} - \frac{\mu^2}{g_{N+1}^2 v_{N+2}^2} s \right) T_N^1(p^2) - \frac{v_{N+1}^2}{v_{N+2}^2} s^2 T_{N-1}^1(s), \quad (6.23)$$

subject to the boundary conditions

$$R_0^1(s) = 0, \quad R_1^1(s) = \text{constant}, \quad S_{-1}^1(s) = 0, \quad S_0^1(s) = \text{constant} \quad (6.24)$$

Equations (6.23) and (6.24) follow simply from the fact that the solution F_{N+1}^j satisfying Dirichlet boundary conditions can be written as $F_{N+1}^j = Y_{N+1} J_j - J_{N+1} Y_j$, where J_j and Y_j are any two independent solutions to eq. (6.20). They have exactly the same form, up to local terms, as the recurrence equations and the boundary conditions for the denominators and the numerators of Padé approximants.

In fact, the recurrence relation in Padé approximation contains three sets of coefficients a_N , b_N and c_N , which define the related orthogonal polynomial. On the deconstruction side we dispose only of two sets: g_N and v_N . However, since rescaling of the numerator and the denominator by the same function does not change the polarization function, Migdal approximation and deconstruction are equivalent if the recurrence equations can be brought to the same form after rescaling T_N^1 by an arbitrary function, $T_N^1 \rightarrow h_N T_N^1$. Taking this into account leads to the norm-independent consistency conditions:

$$\begin{aligned} \frac{g_{N+1}^2 (v_{N+2}^2 + v_{N+1}^2)}{\mu^2} &= -1 - \frac{a_N}{b_N} \\ \frac{g_N^2 g_{N+1}^2 v_{N+1}^4}{\mu^4} &= \frac{c_N}{b_N b_{N-1}} \end{aligned} \quad (6.25)$$

Thus, given a_N , b_N and c_N defining the orthogonal polynomial corresponding to the Padé approximants, we are able to reconstruct the (tilted) deconstruction model that would give exactly the same polarization function. Furthermore, using the dictionary (6.15) we can find the continuum background.

Let us now investigate what deconstruction model corresponds to the Padé approximants found in section 2. Those examples were all associated with Jacobi polynomials

$P_N^{\alpha,\beta}(2s-1)$, whose recurrence relation involves the coefficients

$$\begin{aligned} a_N &= \frac{(2N + \alpha + \beta + 1)(2N + \alpha + \beta + 2)}{(N + 1)(N + \alpha + \beta + 1)}, \\ b_N &= \frac{(2N + \alpha + \beta + 1)(\alpha^2 - \beta^2 - (2N + \alpha + \beta)(2N + \alpha + \beta + 2))}{2(N + 1)(N + \alpha + \beta + 1)(2N + \alpha + \beta)}, \\ c_N &= \frac{(N + \alpha)(N + \beta)(2N + \alpha + \beta + 2)}{(N + 1)(N + \alpha + \beta + 1)(2N + \alpha + \beta)}. \end{aligned} \tag{6.26}$$

For large N the consistency equations can be approximated by:

$$\begin{aligned} \frac{g_{N+1}^2(v_{N+2}^2 + v_{N+1}^2)}{\mu^2} &= 1 + \frac{\alpha^2 - \beta^2}{2} \frac{1}{N^2} + \mathcal{O}(1/N^3), \\ \frac{g_N^2 g_{N+1}^2 v_{N+1}^4}{\mu^4} &= \frac{1}{4} + \frac{1 - 4\beta^2}{16} \frac{1}{N^2} + \mathcal{O}(1/N^3). \end{aligned} \tag{6.27}$$

They can be approximately solved by

$$\begin{aligned} \frac{v_{N+1}^2}{v_N^2} &= 1 + \frac{1 \pm 2\alpha}{N} + \mathcal{O}(1/N^2), \\ \frac{g_{N+1}^2}{g_N^2} &= 1 - \frac{1 \pm 2\alpha}{N} + \mathcal{O}(1/N^2), \\ \frac{g_N^2 v_N^2}{\mu^2} &= \frac{1}{2} \left(1 - \frac{1 \pm 2\alpha}{2N} + \mathcal{O}(1/N^2) \right). \end{aligned} \tag{6.28}$$

The corresponding deconstruction background, at lowest order, does not depend on β . We see that $\alpha = 1/2$ can be matched to flat deconstruction with g_N and v_N independent of N . On the other hand, $\alpha = 1$ is equivalent to deconstruction with

$$v_N^2 \approx v_1^2 \frac{1}{N}, \quad g_N^2 \approx \frac{1}{2} g_1^2 N, \tag{6.29}$$

which by eq. (6.15) is deconstruction of AdS₅ gauge theories latticized in conformal coordinates.

Neumann boundary conditions. The results for the Neumann boundary conditions $F_N^{N+1} = F_N^N$ can be obtained in an analogous way, and below we simply give our results. We define the polynomials:

$$R_N^0(s) = \frac{1}{s-1} s^N (sF_N^1(s) - F_N^0(s)) \quad S_N^0(s) = s^N F_N^0(s) \tag{6.30}$$

Both R_N and S_N are polynomials of degree N in s (one can show that the factor $1/(s-1)$ always cancels out). The polarization function can be written as

$$\Pi(s) = \frac{p^2}{\mu^2} \left(\frac{R_N^0(s)}{S_N^0(s)} + f_0 \right) \tag{6.31}$$

and has the form of a Padé approximant with $J = 0$. The polynomials satisfy a recurrence relation that is different from the Dirichlet case:

$$T_{N+1}^0(s) = \left(\frac{v_{N+1}^2(g_N^2 + g_{N+1}^2) + \mu^2}{g_{N+1}^2 v_{N+2}^2} - \frac{\mu^2}{g_{N+1}^2 v_{N+2}^2} s \right) T_N^0(p^2) - \frac{g_N^2 v_N^2}{g_{N+1} v_{N+2}^2} s^2 T_{N-1}^0(s). \tag{6.32}$$

In obtaining this equation, the correlation between the coefficients of (6.20) is crucial. Adding a mass term would spoil this correlation and the resulting recurrence relation in the Neumann case would not be of the Padé form. This agrees with our discussion of the continuum extra dimensions. The initial conditions are

$$R_0^0(s) = 0, \quad R_1^0(s) = \text{constant}, \quad S_{-1}^0(s) = 0, \quad S_0^0(s) = \text{constant}. \tag{6.33}$$

The recurrence relation and the initial conditions agree with the ones for the Padé numerators and denominators, as long as the consistency conditions

$$\begin{aligned} \frac{v_{N+1}^2(g_{N+1}^2 + g_N^2)}{\mu^2} &= -1 - \frac{a_N}{b_N} \\ \frac{g_N^4 v_N^2 v_{N+1}^2}{\mu^4} &= \frac{c_N}{b_N b_{N-1}} \end{aligned} \tag{6.34}$$

are fulfilled. For Jacobi polynomials on the Migdal side, the large- N approximate solution is given by

$$\begin{aligned} \frac{v_{N+1}^2}{v_N^2} &= 1 - \frac{1 \pm 2\alpha}{N} + \mathcal{O}(1/N^2) \\ \frac{g_{N+1}^2}{g_N^2} &= 1 + \frac{1 \pm 2\alpha}{N} + \mathcal{O}(1/N^2) \\ \frac{g_N^2 v_N^2}{\mu^2} &= \frac{1}{2} \left(1 + \frac{1 \pm 2\alpha}{2N} + \mathcal{O}(1/N^2) \right). \end{aligned} \tag{6.35}$$

We can see that $\alpha = -1/2$ is reproduced by flat deconstruction with g_N and v_N independent of N , while $\alpha = 0$ corresponds again to deconstruction of AdS₅ in conformal coordinates,

$$v_N^2 \approx v_1^2 \frac{1}{N}, \quad g_N^2 \approx \frac{1}{2} g_1^2 N. \tag{6.36}$$

These results are very welcomed, as they show that a single deconstruction setting is able to reproduce the values $l = 0$ and $l = 1$ in the Migdal approximation, just by a change of the IR boundary condition.

7. Conclusions

In this paper we have studied the relations between three different methods of computing correlation functions in strongly-coupled large- N_c theories: the Migdal approach via Padé approximants, the 5D holographic approach via boundary correlators, and the deconstruction approach via external field correlators. We have made explicit the connection between

the Migdal approximation and the other two methods. The key feature of the Padé approximant is that its denominator and numerator can be expressed in terms of some orthogonal polynomial and their associated orthogonal polynomial, respectively. This ensures physical properties of the Padé approximants analogous to those of large- N_c theories. Furthermore, the recurrence relations satisfied by the orthogonal polynomials provide a link with local equations of motion in the other two approaches.

An interesting common feature is that a discrete spectrum is obtained thanks to a violation of quark-hadron duality [20] that is introduced by hand, either by keeping a constant $\tilde{t} = N^2 p^2 / \mu^2$ in the Migdal limit or by cutting off the space with the IR brane. Locality implies that the deviation from the perturbative result in the Euclidean region is exponentially suppressed (oscillating in the Minkowski region). In position space, it corresponds to singularities at a finite distance $\sim z_{\text{IR}}$, which are not seen by the OPE. As discussed in [20, 21], different duality violations, with singularities at infinity, appear in models with linear Regge trajectories, such as the infinite-dimensional holographic model in [22].

We have considered the large- N Migdal limit, in which $N^2 p^2 / \mu^2$ is constant. This forced us to restrict the input functions to the ones that have a conformal form. On the other hand, conformality in the UV is related to asymptotically-AdS₅ spaces. The non-trivial fact that we have explored here is that Migdal's approximation extrapolates this conformal/AdS character all the way down from the UV to the IR, up to an abrupt IR cutoff/brane.⁸ This is related to the particular limit we are considering. It would be interesting to investigate non-conformal input functions, which arise at higher orders of perturbation theory. This would require a different Migdal limit, and it could be speculated that a softer IR cutoff would be generated (possibly involving an infinite extra dimension, as in [23]). Conformality is also broken by power corrections which, in this context, were discussed in [10]. It would also be interesting to study the correspondence for higher-point correlators [24].

The relation between Padé approximation and holographic calculations in 5D or deconstruction could be regarded as a mere mathematical curiosity. However, we expect it to have physical consequences as well. In fact, the Migdal program relies on dispersion relations and is similar in spirit to the SVZ sum rules, which have a solid theoretical basis. The connection with 5D models might shed some light on the unexpected success of AdS/QCD models [6]. Furthermore, Padé approximation is often employed as a unitarization method to extrapolate the predictions of chiral perturbation theory in QCD [25] and in no-Higgs models of electroweak breaking [26] to higher energies. This approach is complementary to Migdal's, for it goes from low to high energies rather than the other way round.⁹ On the

⁸We have also seen that a similar extrapolation is at work in other conformally flat spaces in the massless case.

⁹In [27], it has been pointed out that, because the exact (large- N_c) two-point function is a Stieltjes function, if the chiral contributions to all orders were known, then the large- N limit of the corresponding Padé approximants would exactly reproduce the full two-point function, at arbitrary momenta. In particular, this puts restrictions on the allowed chiral coefficients, directly related to the ones which can be derived from dispersion relations [28].

other hand, alternative unitarization procedures using the notion of extra dimensions have been introduced more recently: the so-called higgsless electroweak breaking [29] and its deconstructed version [30]. Our results suggest that these seemingly unrelated approaches could be equivalent, although to prove it we should study the Padé approximants with a low-energy, rather than high-energy, input.

Acknowledgments

We thank Ian Low and Alex Pomarol for useful discussions. A.F. is partially supported by the European Community Contract MRTN-CT-2004-503369 for the years 2004-2008 and by the MEiN grant 1 P03B 099 29 for the years 2005-2007. M.P.V. is partially supported by MEC (FPA 2003-09298-C02-01) and by Junta de Andalucía (FQM 101 and FQM 437).

A. Derivation of the boundary effective action in deconstruction

We derive here the holographic formula for the two-point correlation function of the “boundary” fields in deconstruction. We work in the tilted deconstruction framework; the minimal deconstruction result can be obtained by setting $\alpha_j = 0$.

The tilted deconstruction action can be rewritten as

$$S = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \left\{ \sum_{j,k=0}^N A_\mu^j D_{\mu\nu}^{jk} A_\mu^k \right\}, \tag{A.1}$$

where the kinetic operator is defined as

$$D_{\mu\nu}^{jk} = (-p^2 \eta_{\mu\nu} + p_\mu p_\nu) \left(\frac{1}{g_j^2} \delta_{j,k} + \frac{\alpha_j}{g_j^2} \delta_{j-1,k} + \frac{\alpha_{j+1}}{g_{j+1}^2} \delta_{j+1,k} \right) + \eta_{\mu\nu} \left((v_j^2 + v_{j+1}^2) \delta_{j,k} - v_j^2 \delta_{j-1,k} - v_{j+1}^2 \delta_{j+1,k} \right). \tag{A.2}$$

For $D_{\mu\nu}^{00}$, $v_0 \equiv 0$ is understood. Our objective is to obtain an effective action for the boundary field A_μ^0 after integrating out all resonances A_μ^j with $j \geq 1$. At tree-level, this can be achieved by solving the equations of motion for the resonances with the boundary background field switched on,

$$\sum_{k=0}^N D_{\mu\nu}^{jk} A_\nu^k = 0 \quad j \geq 1, \tag{A.3}$$

and inserting the solution back into the action (A.1). The solution can be written as

$$A_\mu^j = \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{F_N^j(p^2)}{F_N^0(p^2)} + \frac{p_\mu p_\nu}{p^2} \frac{F_N^j(0)}{F_N^0(0)}, \tag{A.4}$$

where $F_N^j(p^2)$ solves the recurrence relation in j ,

$$\left(v_{j+1}^2 + v_j^2 - \frac{p^2}{g_j^2} \right) F_N^j - \left(v_j^2 + \frac{p^2 \alpha_j}{g_j^2} \right) F_N^{j-1} - \left(v_{j+1}^2 + \frac{p^2 \alpha_{j+1}}{g_{j+1}^2} \right) F_N^{j+1} = 0, \tag{A.5}$$

subject to appropriate boundary conditions: $F_N^N = 0$ in the Dirichlet case and $F_N^{N+1} = F_N^N$ in the Neumann case. Inserting the solution back into the action we obtain

$$S_{\text{eff}} = \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} v_1^2 \left\{ \left(-\eta_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} \right) A_\mu^0 \Pi(p^2) A_\nu^0 + \frac{p_\mu p_\nu}{p^2} A_\mu^0 A_\nu^0 \frac{F_N^1(0) - F_N^0(0)}{F_N^0(0)} \right\}. \quad (\text{A.6})$$

The second term vanishes in the Neumann case, while in the Dirichlet case it is cancelled by tree-level exchange of a massless physical Goldstone boson. Finally, the polarization operator is given by

$$\Pi(p^2) = \frac{F_N^1(p^2)}{F_N^0(p^2)} \left(1 + \frac{\alpha_1}{g_1^2 v_1^2} p^2 \right) + \frac{1}{g_0^2 v_1^2} p^2 - 1. \quad (\text{A.7})$$

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